# Bisimulation Quantifiers and Uniform Interpolation for Guarded First Order Logic

Giovanna D'Agostino, Giacomo Lenzi[a,b]

[a] *University of Udine, Italy, Department of Mathematics and Computer Science, giovanna.dagostino@uniud.it*
[b] *University of Salerno, Italy, Department of Mathematics, gilenzi@unisa.it*

**Abstract**

The idea that the good model-theoretic and algorithmic properties of Modal Logics are due to the guarded nature of their quantification was put forward by Andreka, van Benthem and Nemeti in a series of papers in the '90s, exploiting the satisfiability problem, the tree model property, and other similar properties of the guarded fragment of First Order Logic ($GF$).
Since then, further work on the Guarded Fragment has been done by various authors, in some cases reinforcing this idea, in some others not. At least at first sight, Craig interpolation is on the negative side: there are implications in $GF$ without an interpolant in $GF$, while Modal Logic (and even the $\mu$-calculus, a powerful extension of Modal Logic) enjoys a much stronger form of interpolation, the uniform one, in which the interpolant of a valid implication not only exists, but only depends on the antecedent and on the common language of antecedent and consequent. However, Hoogland and Marx proved that Craig interpolation is restored in $GF$ if we consider the modal character of $GF$ with more attention, that is, if relations appearing on guards are viewed as "modalities" and the rest as "propositions", and only the latter enter in the common language. In this paper we strengthen this result by showing that $GF$ enjoys a Modal Uniform Interpolation Theorem (in the sense of Hoogland and Marx).

*Keywords:*
Bisimulation Quantifiers, Uniform Interpolation, Guarded First Order Logic

## 1. Introduction

The Craig Interpolation property has always been considered as a yardstick to measure the good interplay between syntax and semantics of a logic. In this paper we consider the Guarded Fragment of First Order Logic ($GF$) and show that it inherits from Modal Logic its good behavior with respect to interpolation, provided the modal aspect of $GF$ is considered seriously. This approach has been already undertaken by Hoogland and Marx [1] and results in a proof of Modal Interpolation for the guarded fragment.

In this paper we consider the uniform version of Modal Interpolation, where the interpolant of a valid implication only depends on the antecedent and on the common language of antecedent and consequent. The importance of uniform interpolation can be seen in terms of modularization. Suppose we have a specification of a process in the form of a formula $\phi$ but we are only interested in a particular subset of the language of $\phi$. Then we would like to extract a formula $\psi$ that only deals with this sublanguage, yet is equivalent to $\phi$ as far as this sublanguage is concerned (i.e. a module for this subtask). Uniform interpolation tells us that we can always find such a formula $\psi$.

In contrast with interpolation, the uniform version is not a very robust property when we leave the language fixed but we restrict the class of models: uniform interpolation is satisfied in $K$ and $GL$, but not on logics over transitive models such as $S4$ and $K4$ (although these logics do satisfy interpolation). However, uniform interpolation seems to be more robust with respect to certain language extensions, as in the $\mu$-calculus, where we leave the class of models unchanged but we add some fixed point operators to the language (see [2, 3]).

In this paper, by using the notion of bisimulation quantifiers for $GF$, we prove the Uniform version of Modal Interpolation for this logic.

## 2. Preliminaries

### 2.1. Syntax and Semantics of Guarded First Order Logic

**Definition 2.1.** *Let $\tau$ a vocabulary consisting of a finite number of relational symbols. The* guarded fragment *$GF(\tau)$ of first-order logic is defined inductively as follows:*

1. *every atomic formula $x_i = x_j$ or $r(x_{i_1}, \ldots, x_{i_n})$ with $r \in \tau$ belongs to $GF(\tau)$;*
2. *$GF(\tau)$ is closed under boolean operations;*

3. *if $\psi$ is a formula of $GF(\tau)$, $\mathbf{x}, \mathbf{y}$ are tuples of variables, $\alpha$ is an atomic formula with $free(\psi) \subseteq free(\alpha) = \mathbf{x}, \mathbf{y}$ then*

$$\exists \mathbf{y}\ (\alpha(\mathbf{x}, \mathbf{y}) \wedge \psi) \qquad \textit{and} \qquad \forall \mathbf{y}\ (\alpha(\mathbf{x}, \mathbf{y}) \rightarrow \psi)$$

*belong to $GF(\tau)$ ($\alpha$ is called the* guard *of the formula).*

As usual, we may suppose that formulas are in negation normal form, that is, negation appears only in front of atomic formulas. The free variables $free(\phi)$ of a formula $\phi$ are defined as usual.

A formula $\phi \in GF(\tau)$ inherits its semantics from first order logic: if $\mathbf{a} = a_1, \ldots, a_h$ is a tuple of elements in a first order structure $\mathfrak{A}$ for the language $\tau$, then $\mathfrak{A}, \mathbf{a} \models \phi$ has the usual meaning (where we implicitly suppose that the free variables $free(\phi)$ of $\phi$ are among $x_1, \ldots, x_h$ and are interpreted as the corresponding elements in the tuple $\mathbf{a}$).

If $\phi, \psi \in GF(\tau)$ and $free(\phi) \cup free(\psi) \subseteq \mathbf{x}$, we write $\phi \models \psi$ if and only if for all $\mathfrak{A}, \mathbf{a}$ with $\mathfrak{A}, \mathbf{a} \models \phi$ we have $\mathfrak{A}, \mathbf{a} \models \psi$.

Given a formula $\phi \in GF(\tau)$, let $\mathcal{L}(\phi)$ be the set of relational symbols appearing in $\phi$, and let $Guard(\phi)$ be the set of relational symbols having at least one occurrence as a guard in $\phi$.
If $\tau' \subseteq \tau$, we define the fragment

$$GF(\tau', \tau) = \{\phi \in GF(\tau) : Guard(\phi) \subseteq \tau'\}.$$

If $\phi \in GF(\tau', \tau)$, its *quantification rank* $qr(\phi)$ is defined inductively as usual:

1. if $\phi$ is atomic then $qr(\phi) = 0$.
2. $qr(\neg\phi) = qr(\phi)$, $qr(\phi \vee \psi) = max\{qr(\phi), qr(\psi)\}$;
3. $$qr(\exists \mathbf{y}\ (\alpha(\mathbf{x}, \mathbf{y}) \wedge \psi)) = qr(\forall \mathbf{y}\ (\alpha(\mathbf{x}, \mathbf{y}) \rightarrow \psi)) = qr(\psi) + 1.$$

Given a structure $\mathfrak{A}$ and a tuple $\mathbf{a} = (a_1, \ldots, a_h)$, we denote by $set(\mathbf{a})$ the set $\{a_1, \ldots, a_h\}$. A tuple $\mathbf{a} = (a_1, \ldots, a_h)$ is said to be $r$-*guarded* if $\mathfrak{A} \models r(a_1, \ldots, a_h)$. The family of $r$ guarded tuples of $\mathfrak{A}$ is denoted by $Guard_r(\mathfrak{A})$. The family of all $r$ guarded tuples of $\mathfrak{A}$ with $r \in \tau'$ is denoted by $Guard_{\tau'}(\mathfrak{A})$. A set $K$ is said to be $r$-guarded if there is an $r$-guarded tuple $\mathbf{a}$ such that $K \subseteq set(\mathbf{a})$; we still write $K \in Guard_{\tau'}(\mathfrak{A})$ for an $r$-guarded set $K$ with $r \in \tau'$.

A $(\tau', \tau)$-*structure* is a pair $(\mathfrak{A}, \mathbf{a})$ where $\mathfrak{A}$ is a $\tau$-structure and $\mathbf{a}$ is a guarded tuple in $Guard_{\tau'}(\mathfrak{A})$. $\mathbf{a}$ is called the $\tau'$-*source* of the structure.
We denote by $A, B, \ldots$ the domains of the structures $\mathfrak{A}, \mathfrak{B}, \ldots$, respectively.

**Definition 2.2.** *A partial isomorphism between* $(\tau', \tau)$*-structure* $(\mathfrak{A}, \mathbf{a})$*,* $(\mathfrak{B}, \mathbf{b})$ *is a bijection* $f$ *from set*$(\mathbf{a})$ *to set*$(\mathbf{b})$ *satisfying*

$$(a_{i_1}, \dots, a_{i_{|r|}}) \in r^{\mathfrak{A}} \Leftrightarrow (fa_{i_1}, \dots, fa_{i_{|r|}}) \in r^{\mathfrak{B}},$$

*for all* $r \in \tau$ *and* $\{a_{i_1}, \dots, a_{i_{|r|}}\} \subseteq set(\mathbf{a})$*.*

If $\mathbf{a} = (a_1, \dots, a_k)$ and $\mathbf{b} = (b_1, \dots, b_k)$ are tuples of the same length in $A, B$, respectively, we denote the correspondence $\{(a_i, b_i) : i \in \{1, \dots, k\}\}$ by $\mathbf{a} \mapsto \mathbf{b}$.

*2.2. Bisimulation*

The following definition (is a generalization to our context of a definition that) can be found in [4].

**Definition 2.3.** *A* guarded bisimulation *between two* $(\tau', \tau)$*-structures* $(\mathfrak{A}, \mathbf{a})$*,* $(\mathfrak{B}, \mathbf{b})$ *with domains* $A, B$ *respectively, is a set* $I$ *of finite partial* $\tau$*-isomorphisms from* $\tau'$*-guarded sequences in* $\mathfrak{A}$ *to* $\tau'$*-guarded sequences in* $\mathfrak{B}$ *such that* $\mathbf{a} \mapsto \mathbf{b} \in I$*, and for all* $f = \mathbf{c} \mapsto \mathbf{d} \in I$ *it holds:*

1. *Forth: for every guarded tuple* $\mathbf{c}' \in Guard_{\tau'}(\mathfrak{A})$ *there exists a partial isomorphism* $g = \mathbf{c}' \mapsto \mathbf{d}'$ *in* $I$ *such that* $f, g$ *agree on the elements occurring in* $set(\mathbf{c}) \cap set(\mathbf{c}')$*;*

2. *Back: for every tuple* $\mathbf{d}' \in Guard_{\tau'}(\mathfrak{B})$ *there exists a partial isomorphism* $g = \mathbf{c}' \mapsto \mathbf{d}'$ *in* $I$ *such that* $f^{-1}, g^{-1}$ *agree on the elements occurring in* $set(\mathbf{d}) \cap set(\mathbf{d}')$*.*

If there exists a bisimulation between two $(\tau', \tau)$-structures $(\mathfrak{A}, \mathbf{a})$, $(\mathfrak{B}, \mathbf{b})$ we write

$$(\mathfrak{A}, \mathbf{a}) \sim^{(\tau', \tau)} (\mathfrak{B}, \mathbf{b}).$$

**Proposition 2.4.** *If* $\phi$ *is a formula in* $GF(\tau', \tau)$ *and* $(\mathfrak{A}, \mathbf{a}), (\mathfrak{B}, \mathbf{b})$ *are* $(\tau', \tau)$ *structure such that* $(\mathfrak{A}, \mathbf{a}) \sim^{(\tau', \tau)} (\mathfrak{B}, \mathbf{b})$ *then*

$$(\mathfrak{A}, \mathbf{a}) \models \phi \Leftrightarrow (\mathfrak{B}, \mathbf{b}) \models \phi$$

As for modal logic, we have a bounded variant of bisimulation:

**Definition 2.5.** *A $n$-*bounded guarded bisimulation *between two $(\tau', \tau)$ structures $(\mathfrak{A}, \mathbf{a}), (\mathfrak{B}, \mathbf{b})$ with domains $A, B$ respectively, is a tuple $(I_0, \ldots, I_n)$ of finite partial $\tau$-isomorphisms from $\tau'$-guarded sequences in $\mathfrak{A}$ to $\tau'$-guarded sequences $\mathfrak{B}$ such that $\mathbf{a} \mapsto \mathbf{b} \in I_n$, and for all $i \in \{0, \ldots, n-1\}$ and $f = \mathbf{c} \mapsto \mathbf{d} \in I_{i+1}$ it holds:*

1. *Forth: for every guarded tuple $\mathbf{c}' \in Guard_{\tau'}(\mathfrak{A})$ there exists a partial isomorphism $g = \mathbf{c}' \mapsto \mathbf{d}'$ in $I_i$ such that $f, g$ agree on the elements occurring in $set(\mathbf{c}) \cap set(\mathbf{c}')$;*

2. *Back: for every tuple $\mathbf{d}' \in Guard_{\tau'}(\mathfrak{B})$ there exists a partial isomorphism $g = \mathbf{c}' \mapsto \mathbf{d}'$ in $I_i$ such that $f^{-1}, g^{-1}$ agree on the elements occurring in $set(\mathbf{d}) \cap set(\mathbf{d}')$.*

The existence of an $n$-bounded guarded bisimulation between the $(\tau', \tau)$-structures $(\mathfrak{A}, \mathbf{a}), (\mathfrak{B}, \mathbf{b})$ is denoted by

$$(\mathfrak{A}, \mathbf{a}) \sim_n^{(\tau', \tau)} (\mathfrak{B}, \mathbf{b})$$

.

**Proposition 2.6.** *If $\phi$ is a formula in $GF(\tau', \tau)$ with quantification rank smaller than $n$ and $(\mathfrak{A}, \mathbf{a}) \sim_n^{(\tau', \tau)} (\mathfrak{B}, \mathbf{b})$ then*

$$(\mathfrak{A}, \mathbf{a}) \models \phi \Leftrightarrow (\mathfrak{B}, \mathbf{b}) \models \phi$$

If $\tau' \subseteq \sigma \subseteq \tau$ and $(\mathfrak{A}, \mathbf{a})$ is a $(\tau', \tau)$ structure, we denote by $(\mathfrak{A}, \mathbf{a})_{|\sigma}$ the $(\tau', \sigma)$ structure obtained from $(\mathfrak{A}, \mathbf{a})$ by considering only relations in $\sigma$. Moreover, if $(\mathfrak{B}, \mathbf{b})$ is another $(\tau', \tau)$ structure we write

$$(\mathfrak{A}, \mathbf{a}) \sim^{(\tau', \sigma)} (\mathfrak{B}, \mathbf{b})$$

for

$$(\mathfrak{A}, \mathbf{a})_{|\sigma} \sim^{(\tau', \sigma)} (\mathfrak{B}, \mathbf{b})_{|\sigma},$$

and the same for bounded bisimulations.

### *2.3. Forms of Interpolation*

In this section we list various forms of interpolation we shall consider for the Guarded Fragment.

**Definition 2.7. (Craig Interpolant)**
*Suppose $\phi, \psi \in GF$ are such that $\phi \models \psi$. A formula $\theta \in L(\phi) \cap L(\psi)$ is said to be a* Craig Interpolant *for the implication if $\phi \models \theta$ and $\theta \models \psi$.*

**Example 2.8.** *It is well known that Craig Interpolation fails for $GF$, that is, not all valid implications in $GF$ have a Craig Interpolant. We report here a counterexample from [8]. Let*

$$\phi(x) := \exists y \exists z (G(x, y, z) \wedge R(x, y) \wedge R(y, z) \wedge R(z, x)),$$

$$\psi(x) := [P_0(x) \wedge \bigwedge_{i=0}^{2} \forall y \forall z (R(y, z) \to (P_i(y) \to P_{i+1}(z)))] \to P_3(x).$$

*We have $\phi \models \psi$ but it is easily seen that an interpolant $\theta$ in the language $L(\phi) \cap L(\psi) = \{R\}$ would be equivalent to $\exists y \exists z (R(x, y) \wedge R(y, z) \wedge R(z, x))$; however the property expressed by the latter formula is not invariant under guarded bisimulation, hence there is no guarded interpolant for the implication $\phi \models \psi$.*

In this example we use formulas with one free variable. In [1] a valid implication between *sentences* of $GF$ without an interpolant in $GF$ is proposed (but the example is slightly more complicate).

To restore interpolation one could consider more expressive fragments of First Order Logic where inteprolation holds, like the Guarded Negation Fragment (see [7]). On the other hand, if we want to keep the fragment unchanged, in [1] it is shown that we should take the modal character of $GF$ more seriously, that is, we should consider relations appearing on guards as *modalities* and the rest as *propositions*, where only the latter enter in the common language. In our terminology, we consider the language $GF(\tau', \tau)$, and we divide all relations $r \in \tau$ between *modalities* (if $r \in \tau'$), and *propositions* (if $r \in \tau \setminus \tau'$).

**Theorem 2.9. (Modal Interpolation for $GF(\tau', \tau)$ [1])**
*Let $\tau', \tau, \sigma$ be finite vocabularies with $\tau' \subseteq \tau, \tau' \subseteq \sigma$, and $\phi \in GF(\tau', \tau), \psi \in GF(\tau', \sigma)$. Then: $\phi \models \psi \Rightarrow \exists \theta \in GF(\tau', \tau \cap \sigma)$ with $\phi \models \theta$ and $\theta \models \psi$.*

We now see that 2.8 is no longer a counterexample for Modal Interpolation in $GF(\tau', \tau)$, simply because the only way to obtain $\phi \in GF(\tau', \tau)$ and $\psi \in GF(\tau', \sigma)$ is to consider $\tau' = \{G, R\}$, $\tau = \{G, R\}$, $\sigma = \{G, R, P_0, P_1, P_2, P_3\}$, and in this case the interpolant $\theta \in GF(\tau', \tau \cap \sigma) = GF(\{G, R\}, \{G, R\})$ for the implication $\phi \models \psi$ is simply $\theta = \phi$.

Having the Modal example in mind we consider a stronger form of interpolation, where the interpolant of an implication $\phi \models \psi$ does not depend on $\psi$ but only on the *common language* of $\phi$ and $\psi$. As in Theorem 2.9, we have to distinguish between propositions and modalities, but once this is done we can state:

**Theorem 2.10. (Uniform Modal Interpolation for $GF(\tau', \tau)$)** *Let $\tau', \tau, \sigma$ be finite vocabularies with $\tau' \subseteq \sigma \subseteq \tau$. For any formula $\phi \in Guard(\tau', \tau)$ there exists a formula $\theta \in Guard(\tau', \sigma)$ such that*

1. $\models \phi \to \theta$;
2. *if $\psi \in Guard(\tau', \nu)$ for a vocabulary $\nu$ such that $\tau' \subseteq \nu$, $\tau \cap \nu \subseteq \sigma$, and $\models \phi \to \psi$, then $\models \theta \to \psi$.*

We will prove this theorem using techniques from Modal Logic, such as bisimulation, bounded bisimulations, modal type, unravelings, bisimulation quantifiers, reinforcing the idea that the right way to look at the guarded fragment is to distinguish between propositions and modalities, at least when interpolation is concerned.

## 3. Trees and Unravelings

In this paragraph we fix the finite relational vocabularies $\tau' \subseteq \tau$ and we denote by $k = k(\tau')$ the maximal arity of $\tau'$ relations.

**Definition 3.1.** *A $\Sigma_{\tau',\tau}$-tree is a pair $(T, \mathcal{T})$ where $T$ is a tree and $\mathcal{T}$ is a labeling of vertices and edges of $T$ such that:*

1. *$\mathcal{T}(v)$ consists of a $\tau$-structure over a domain $T(v) = \{i_1, \ldots, i_h\} \subseteq \{1, \ldots, k(\tau')\}$, such that $(i_1, \ldots, i_h)$ is $\tau'$- guarded in $\mathcal{T}(v)$;*
2. *$\mathcal{T}(v, v') \subseteq \{1, \ldots, k(\tau')\}$, for all edges $(v, v')$ in $T$.*

*A $\Sigma_{\tau',\tau}$-tree $(T, \mathcal{T})$ is said to be* consistent *if, whenever $(v, v')$ is an edge in $T$ then the domains $T(v), T(v')$ of the $\tau$ structures $\mathcal{T}(v), \mathcal{T}(v')$ contain $\mathcal{T}(v, v')$ and $\mathcal{T}(v), \mathcal{T}(v')$ agree over $\mathcal{T}(v, v')$. In other words, $\mathcal{T}(v, v')$ asserts which elements are the same in $T(v)$ and $T(v')$, and the $\tau$-structures in $v$ and $v'$ restricted to that set of elements must match.*

We extend the function $\mathcal{T}$ from edges of $T$ to pairs $(v, v') \in T \times T$ by letting $\mathcal{T}(v, v')$ to be the set of all $i$ belonging to all edges in the simple undirected path from $v$ to $v'$.

The levels $T_h$ of a $\Sigma_{\tau',\tau}$-tree $(T, \mathcal{T})$ are defined inductively, as usual: $T_0$ is the set containing only the root of $T$; $T_{h+1}$ contains all sons of nodes in $T_h$.

We next define some special kind of $\Sigma_{\tau',\tau}$ consistent trees, which are the analogues of *unravelings* in modal logic.

**Definition 3.2.** *(see e.g. [5]) An* unraveling *of a $(\tau', \tau)$ structure $(\mathfrak{A}, \mathbf{a})$ is a $\Sigma_{\tau',\tau}$ consistent tree $(U, \mathcal{U})$ satisfying the following.*

1. *The domain $U$ is given by all finite sequences $t = \mathbf{a}^0\mathbf{a}^1\mathbf{a}^2 \ldots \mathbf{a}^m$ with $\mathbf{a}^i \in Guard_{\tau'}(\mathfrak{A})$, and $\mathbf{a}^0 = \mathbf{a}$. We also define $end(t) := \mathbf{a}^m$.*
2. *For every $t = \mathbf{a}^0\mathbf{a}^1\mathbf{a}^2 \ldots \mathbf{a}^m$ in $U$, the sons of $t$ in $U$ are all the sequences of the form $t\mathbf{a}$ with $\mathbf{a} \in Guard_{\tau'}(\mathfrak{A})$.*
3. *For every $t = \mathbf{a}^0\mathbf{a}^1\mathbf{a}^2 \ldots \mathbf{a}^m$ in $U$, there exists a $\tau$- isomorphism $\pi_t : \mathfrak{A}_{|set(\mathbf{a}^m)} \to \mathfrak{U}_t$ where $\mathfrak{U}_t$ is a $\tau$-structure with domain contained in $\{1, \ldots k(\tau')\}$; we also require that if $end(t) = \mathbf{a} = (a_1 \ldots, a_h)$, $end(t') = \mathbf{a}'$, and $t' \in Son(t)$ then $\pi_t, \pi_{t'}$ agree on $set(\mathbf{a}) \cap set(\mathbf{a}')$; we define $\mathcal{U}(t)$ to be the $\tau$ structure based on the $\tau'$-guarded sequence*

   $$(\pi_t(a_1), \ldots, \pi_t(a_h)).$$

   *Moreover, we let*

   $$\mathcal{U}(t, t') := \{\pi_t(a) : a \in set(\mathbf{a}) \cap set(\mathbf{a}')\}.$$

Given a $\Sigma_{\tau',\tau}$ consistent tree $(T, \mathcal{T})$, we denote by $rec(T, \mathcal{T})$ the $(\tau', \tau)$-structure *recovered* from $\mathcal{T}$, which is defined as follows. We first consider the disjoint sum of $\mathcal{T}$ node labels:

$$B = \bigcup \{T(v) \times \{v\} : v \in T\}.$$

Let $\approx$ be the least equivalence relation on $B$ this such that

$$(i, v) \approx (i, v') \text{ if } (v, v') \text{ is an edge in } T \text{ and } i \in \mathcal{T}(v, v').$$

Let $[i, v]$ be the $\approx$ equivalence class of $(i, v)$.

Notice that, since $T$ is a tree, for all $(i_1, v_1), \ldots, (i_h, v_h)$ and $r \in \tau$ the following properties are equivalent:

$$\exists v \; (i_1, v_1) \approx (i_1, v) \ldots, (i_h, v_h) \approx (i_h, v) \text{ and } \mathcal{T}(v) \models r(i_1, \ldots, i_h);$$

$$\forall v \text{ if } (i_1, v_1) \approx (i_1, v) \ldots, (i_h, v_h) \approx (i_h, v) \text{ then } \mathcal{T}(v) \models r(i_1, \ldots, i_h).$$

Finally:

**Definition 3.3.** *The $\tau$-structure $rec(T, \mathcal{T})$ is defined as follows:*

1. *its domain is the set $B/\approx$;*
2. *if $r \in \tau$ then*

$$r^{B/\approx}([i_1, v_1], \ldots, [i_h, v_h])$$

$$\Updownarrow$$

$$\exists v\ (i_1, v_1) \approx (i_1, v) \ldots, (i_h, v_h) \approx (i_h, v) \text{ and } \mathcal{T}(v) \models r(i_1, \ldots, i_h);$$

**Lemma 3.4.** *A tuple $([i_1, v_1], \ldots, [i_h, v_h])$ is $r$-guarded in $rec(T, \mathcal{T})$ if and only if there exists a $v$ such that $(i_1, v_1) \approx (i_1, v) \ldots, (i_h, v_h) \approx (i_h, v)$ and $\{i_1, \ldots, i_h\}$ is $r$-guarded in $\mathcal{T}(v)$. Given two guarded tuples*

$$([i_1, v], \ldots, [i_h, v]), \quad ([j_1, w], \ldots, [j_l, w])$$

*the set of elements occurring in both tuples is:*

$$\{[i, v] : i \in \{i_1, \ldots, i_h\} \cap \{j_1, \ldots, j_l\} \cap \mathcal{T}(v, w)\}.$$

**Definition 3.5.** *The $(\tau', \tau)$ structure recovered from $(T, \mathcal{T})$ consists of the $\tau$-structure $rec(T, \mathcal{T})$ and the source $([i_1, r] \ldots, [i_h, r])$, where $r$ is the root of $T$ and $(i_1, \ldots, i_h)$ is the source of the $(\tau', \tau)$-structure $\mathcal{T}(r)$.*

**Proposition 3.6.** *If $(\mathfrak{A}, \mathbf{a})$ is a $(\tau', \tau)$-structure and $(\mathfrak{B}, \mathbf{b})$ is the structure recovered from a $(\tau', \tau)$ unraveling $(U, \mathcal{U})$ of $(\mathfrak{A}, \mathbf{a})$, then :*

$$(\mathfrak{A}, \mathbf{a}) \sim^{(\tau', \tau)} (\mathfrak{B}, \mathbf{b}).$$

*In particular, if $\phi(\overline{x})$ is a formula in $Guard(\tau', \tau)$ then*

$$(\mathfrak{A}, \mathbf{a}) \models \phi \Leftrightarrow (\mathfrak{B}, \mathbf{b}) \models \phi.$$

*Proof.* The bisimulation is given by all partial isomorphisms of the form

$$(c_1, \ldots, c_n) \mapsto ([\pi_v(c_1), v], \ldots, [\pi_v(c_n), v])$$

where $\mathbf{c} = (c_1, \ldots, c_n) \in Guard_{\tau'}(\mathfrak{A})$ and $end(v) = \mathbf{c}$. □

We can transfer the notion of a bisimulation from structures to $\Sigma_{\tau', \tau}$ trees:

**Definition 3.7.** *A* guarded tree bisimulation *between two $\Sigma_{\tau',\tau}$ trees $(T, \mathcal{T})$, $(S, \mathcal{S})$ with roots $r_T$, $r_S$, respectively, consists of a relation $B \subseteq T \times S$ containing $(r_T, r_S)$ such that, for each $(t, s) \in B$ there exists a partial $\tau$-isomorphism $f_{t,s} : \mathcal{T}(t) \to \mathcal{S}(s)$ with the following properties:*

1. *Forth: for every son $t'$ of $t$ there exists a son $s'$ of $s$ with $(t', s') \in B$ such that $f_{t,s}, f_{t',s'}$ agree on the elements occurring in $\mathcal{T}(t, t')$, and $f_{t,s}(\mathcal{T}(t, t')) \subseteq \mathcal{S}(s, s')$;*

2. *Back: for every son $s'$ of $s$ there exists a son $t'$ of $t$ with $(t', s') \in B$ such that $f^{-1}_{t,s}, f^{-1}_{t',s'}$ agree on the elements occurring in $\mathcal{S}(s, s')$, and $f^{-1}_{s,t}(\mathcal{S}(s, s')) \subseteq \mathcal{T}(t, t')$.*

As before, we can give a similar definition for bounded $n$-bisimulation, using a sequence $(B_0, \ldots, B_n)$ of relations and going from pairs in $B_i$ to pairs in $B_{i-1}$.

**Remark 3.8.** *If $B = (B_n, \ldots, B_0)$ is a bounded guarded bisimulation between $\Sigma_{\tau',\tau}$ trees $(T, \mathcal{T})$, $(S, \mathcal{S})$, we may suppose without loss of generality that if $(t, s) \in B_h$ then:*

1. *$t \in T_{n-h}$ and $s \in S_{n-h}$;*
2. *if $t$ is not the root of $T$ then $s$ is not the root of $S$; if $t^{-1}, s^{-1}$ are the fathers of $t, s$, respectively, then $(t^{-1}, s^{-1}) \in B_{h+1}$ and either*

$$f_{t,s}(\mathcal{T}(t^{-1}, t)) \subseteq \mathcal{S}(s^{-1}, s) \qquad \text{or} \qquad (f_{t,s})^{-1}(\mathcal{S}(s^{-1}, s)) \subseteq \mathcal{T}(t^{-1}, t).$$

It is also easy to see that:

**Lemma 3.9.** *Two structures are bisimilar (bounded bisimilar) if and only if the corresponding unravelings are bisimilar (bounded bisimilar).*

*3.1. Copying nodes*

A typical property of Modal Logic is that, given a structure and a pair of father and son nodes, we can obtain a bisimilar structure in which the son has been duplicated. This can be done in the context of $GF$ as well, as follows. Let $(T, \mathcal{T})$ be a consistent $\Sigma_{\tau',\tau}$-tree. Fix a node $u \in T$ different from the root, and a subset $A \subseteq \mathcal{T}(u^{-1}, u)$, where $u^{-1}$ is the father of $u$. We define a new $\Sigma_{\tau',\tau}$-tree $(T, \mathcal{T})_{u,A}$ as follows:

- the tree $T'$ is obtained from $T$ by adding to the tree $T$ a copy $(T_u)' = \{u' : u \in T\}$ of the subtree $T_u$ rooted in $u$, in such a way that the copy $u'$ of $u$ is now a new son of $u^{-1}$;

- $\mathcal{T}'(w) = \begin{cases} \mathcal{T}(w) \text{ if } w \in T; \\ \mathcal{T}(t) \text{ if } w = t' \text{ is a new node which is a copy of a node } t \in T_u; \end{cases}$

- $\mathcal{T}'(w,z) = \begin{cases} \mathcal{T}(w,z) \text{ if } w,z \in T; \\ \mathcal{T}(s,t) \text{ if } w = s', z = t' \text{ are copies of the nodes } s,t \in T_u; \\ A \text{ if } w = u^{-1} \text{ and } z = u' \text{ is the copy of the node } u \in T. \end{cases}$

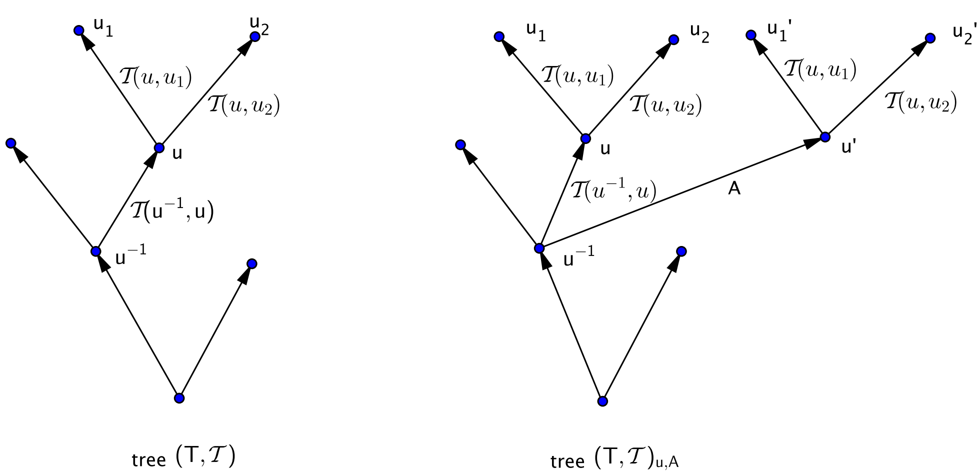


Figure 1: copying a son

In this way, among the sons of $u^{-1}$ in $T'$ we now have both $u$ and its copy $u'$. Notice that, during the copying, we are allowing $A \subseteq \mathcal{T}(u^{-1}, u)$, so we are not just duplicating the subtree rooted at $u$; we are also changing the relationship between $u^{-1}$ and the new subtree.

**Lemma 3.10.** $(T, \mathcal{T})_{u,A}$ *is a* $\Sigma_{\tau',\tau}$*-consistent tree which is* $(\tau', \tau)$*-bisimilar to* $(T, \mathcal{T})$.

*Proof.* The $\Sigma_{\tau',\tau}$ tree $(T',\mathcal{T}')_{u,A}$ is clearly consistent. Moreover, the set

$$B = \{(v,v) : v \in T\} \cup \{(t,t') : t \in T_u\},$$

together with $f_{v,v} = id_{\mathcal{T}(v)}, f_{t,t'} = id_{\mathcal{T}(t)}$ (remember that $\mathcal{T}(t') = \mathcal{T}(t)$, by definition) is a bisimulation between $(T,\mathcal{T})$ and $(T,\mathcal{T})_{u,A}$. Here we only check the back property of a bisimulation w.r.t. the pair $(u^{-1},u^{-1}) \in B$ and the new $u$-son $u'$: in this case we have the isomorphism $f_{u,u'} = id_{\mathcal{T}(u)}$ for which it holds:

- $f^{-1}_{u^{-1},u^{-1}}$ coincides with $f^{-1}_{u,u'}$ on $\mathcal{T}(u^{-1},u') = A$, since both isomorphisms are identities;
- $f^{-1}_{u^{-1},u^{-1}}(\mathcal{T}'(u^{-1},u')) = id^{-1}_{\mathcal{T}(u^{-1})}(A) = A \subseteq \mathcal{T}(u^{-1},u)$.

□

Similarly, for each set $X$, we can copy $|X|$-times a node $u$ using new nodes $u_x$ for all $x \in X$ and different sets $A_x$ to label the new edges $(u^{-1},u_x)$, provided $A_x \subseteq \mathcal{T}(u^{-1},u)$; we can also simultaneously copy a set of vertices $V$, where each $u \in V$ has the same level in $T$, using a different copy $u_x$ for all $x$ in a fixed set $X(u)$, and use sets $A_x$ with $x \in X(u)$ to label the new edges $(u^{-1},u_x)$, provided $A_x \subseteq \mathcal{T}(u^{-1},u_x)$.

**Lemma 3.11.** *If two $\Sigma_{\tau',\tau}$-consistent trees $(T,\mathcal{T})$ and $(S,\mathcal{S})$ are $n$-bisimilar via a bisimulation $B_0 \cup \ldots \cup B_n$ then, up to a bisimulation, we may suppose that for all $s \in S_h$, $h \leq n$ there exists a unique pair $(t,s) \in B_0 \cup \ldots \cup B_n$.*

*Proof.* It is enough to duplicate nodes in $(S,\mathcal{S})$, as in Lemma 3.10. By Remark 3.8 we may suppose that if $(t,s) \in B_h$ then $s \in S_{n-h}, t \in T_{n-h}$ and, if $h \leq n$ and $(t,s) \in B_h$, then $(t^{-1},s^{-1}) \in B_{h+1}$ and either $f_{t,s}(\mathcal{T}(t^{-1},t)) \subseteq \mathcal{S}(s^{-1},s)$ or $f^{-1}_{t,s}(\mathcal{S}(s^{-1},s)) \subseteq \mathcal{T}(t^{-1},t)$ (where $t^{-1},s^{-1}$ are the fathers of $t,s$, respectively). Hence, if $r_S$ is the root of $S$ and $(t,r_S) \in B_h$ then $h = n$ and $t = r_T$. Suppose we already achieved the desired property for all nodes at level $h < n$ in the tree $S$, and consider a node $s$ at level $h+1$ such that

$$|\{t \in T : (t,s) \in B_0 \cup \ldots \cup B_n\})| > 1.$$

Denote by $B^{-1}(s)$ the set $\{t : (t,s) \in B_0 \cup \ldots \cup B_n\}$ and let $s^{-1}$ be the father of $s$. By induction, $|B^{-1}(s^{-1})| = 1$. If $t' \in B^{-1}(s^{-1})$, then for every

$t \in B^{-1}(s)$ we have $t^{-1} = t'$. Fix $t_0 \in B^{-1}(s)$ and $t \neq t_0$, $t \in B^{-1}(s)$ and consider the set $A_t$ defined by

$$A_t := \begin{cases} f_{t,s}(\mathcal{T}(t^{-1}, t)) \text{ if } f_{t,s}(\mathcal{T}(t^{-1}, t)) \subseteq \mathcal{S}(s^{-1}, s); \\ \mathcal{S}(s^{-1}, s) \text{ if } f_{t,s}^{-1}(\mathcal{S}(s^{-1}, s)) \subseteq \mathcal{T}(t^{-1}, t). \end{cases}$$

Since $A_t \subseteq \mathcal{S}(s^{-1}, s)$, using Lemma 3.10 we see that the tree $(S, \mathcal{S})_{s,A_t}$ is bisimilar to $(S, \mathcal{S})$. Moreover, we prove that $(S, \mathcal{S})_{s,A_t}$ is $n$-bisimilar to $(T, \mathcal{T})$. If $h + 1 \leq k \leq n$, define $B'_{n-k}$ to be the set $B_{n-k}$ in which all pairs $(u, v)$ and all isomorphisms $f_{u,v}$ with $u \in T_t,\ v \in S_s$ have been substituted with the pairs $(u, v')$ and with the isomorphism $f_{u,v'} := f_{u,v}$, where $v'$ is the copy of $v$ in $(S_s)'$.

If $B' = (B'_0, B'_1, \ldots, B'_{n-h-1}, B_{n-h}, B_{n-h+1}, \ldots, B_n)$ then one can check that $B'$ is an $n$-bisimulation between $(T, \mathcal{T})$ and $(S, \mathcal{S})_{s,A_t}$. In this way the pair $(t, s) \in B_{n-h-1}$ has been substituted with $(t, s') \in B'_{n-h-1}$.

Similarly, we may substitute all pairs $(t, s) \in B_h$ with $t \neq t_0$, simultaneously for all nodes $s \in S_{h+1}$, obtaining a new tree in which the property stated in the Lemma is true for all nodes up to level $h + 1$. The Lemma is proved by proceeding in this way up to level $n$.

□

Finally, in proving interpolation we shall need to expand a language with new predicate symbols.

**Definition 3.12.** *Let $\tau \subseteq \sigma$ be two finite relational vocabulary. We say that a $\Sigma_{\tau',\sigma}$ tree $(S, \mathcal{S})$ is a $\sigma$-*expansion *of a $\Sigma_{\tau',\tau}$ tree $(T, \mathcal{T})$ if $S = T$ and $\mathcal{T}$ is obtained from $\mathcal{S}$ by just restricting the labels of the vertices (which in $\mathcal{S}$ are $\sigma$-structures) to the vocabulary $\tau$. In symbols*

$$\mathcal{S}(v)_{|\tau} = \mathcal{T}(v), \forall v \in S$$

**Lemma 3.13.** *If $(S, \mathcal{S})$ is a consistent $\sigma$-expansion of a $\Sigma_{\tau',\tau}$ tree $(T, \mathcal{T})$ then the $\sigma$-structure $rec(S, \mathcal{S})$, restricted to $\tau$, is isomorphic to $rec(T, \mathcal{T})$.*

## 4. Expanding a bounded bisimulation

This section is devoted to the proof of a key lemma stating that, whenever we have $\tau' \subseteq \tau$ and $p \in \tau \setminus \tau'$ then, up to $(\tau', \tau \setminus \{p\})$-bisimulations, any $(\tau', \tau \setminus \{p\})$ bounded bisimulation can be expanded to a $(\tau', \tau)$ bounded bisimulation (notice that the corresponding result for modal logic $K$ is already known and has been used in [10] to prove Uniform Interpolation for this logic). The reader should compare this result with amalgamation (see Proposition 7.1), which is often used to prove Craig Interpolation. As we shall see this *bounded* form of amalgamation will allow us to prove the closure under bisimulation quantifiers for guarded logics, which, together with the classical amalgamation result for $GF$, implies uniform interpolation.

**Lemma 4.1. (Bounded Amalgamation)**
*Let $(\mathfrak{A}, \mathbf{a})$ be a $(\tau', \tau)$-structure, and $(\mathfrak{B}, \mathbf{b})$ be a $(\tau', \tau \setminus \{p\})$-structure, where $p \notin \tau'$, such that*

$$(\mathfrak{A}, \mathbf{a}) \sim_n^{(\tau', \tau \setminus \{p\})} (\mathfrak{B}, \mathbf{b}).$$

*Then there exists a $(\tau', \tau)$-structure $(\mathfrak{C}, \mathbf{c})$ with*

$$(\mathfrak{C}, \mathbf{c}) \sim_n^{(\tau', \tau)} (\mathfrak{A}, \mathbf{a}), \qquad (\mathfrak{C}, \mathbf{c}) \sim^{(\tau', \tau \setminus \{p\})} (\mathfrak{B}, \mathbf{b})$$

*Proof.* Consider unravelings $(T, \mathcal{T})$ and $(S, \mathcal{S})$ of $(\mathfrak{A}, \mathbf{a})$, $(\mathfrak{B}, \mathbf{b})$, respectively. From the hypothesis and Lemma 3.11 we may suppose that there is an $n$-bounded $(\tau', \tau \setminus \{p\})$-bisimulation $B = (B_0, \ldots, B_n)$ between $(T, \mathcal{T})$, $(S, \mathcal{S})$, connecting the two roots, such that if $s$ belongs to the $h$-level $S_h$ of $S$ for $h \leq n$ there is exactly one isomorphism $f_{t,s} \in B_{n-h}$. We use the $\tau \setminus \{p\}$-isomorphisms in $B$ to construct a $\Sigma_{\tau',\tau}$ tree $(S', \mathcal{S}')$ such that:

1. $(S', \mathcal{S}')$ is a $\tau$-expansion of the $\Sigma_{\tau',\tau\setminus\{p\}}$ tree $(S, \mathcal{S})$, that is: $S' = S$, $\mathcal{S}'(u, v) = \mathcal{S}(u, v)$, for all $u, v \in S$, and the restriction of $\mathcal{S}'(s)$ to the relations in $\tau \setminus \{p\}$ is $\mathcal{S}(s)$, for all $s \in S$;
2. $(T, \mathcal{T}) \sim_n^{(\tau', \tau)} (S', \mathcal{S}')$.

This will prove the Lemma, since $(S', \mathcal{S}')$ is $(\tau', \tau \setminus \{p\})$ bisimilar to $(S, \mathcal{S})$ which in turn is $(\tau', \tau \setminus \{p\})$ bisimilar to $(\mathfrak{B}, \mathbf{b})$; hence the structure $(\mathfrak{C}, \mathbf{c}) := rec(S', \mathcal{S}')$ satisfies the Lemma.

To construct the $\Sigma_{\tau',\tau}$ tree $(S', \mathcal{S}')$ we proceed as follows: consider the (unique) isomorphism $f_{r_T, r_S} : (\mathcal{T}(r_T))_{\tau\setminus\{p\}} \to \mathcal{S}(r_S)$ with $f_{r_T, r_S} \in B_n$ and use $f_{r_T, r_S}$ to copy the interpretation of $p$ from the $\tau$-structure $\mathcal{T}(r_T)$ to the domain of $\mathcal{S}(r_S)$. In this way we obtain a $\tau$-structure $\mathcal{S}'(r_S)$ which is a $\tau$-expansion of the $\tau \setminus \{p\}$ structure $\mathcal{S}(r_S)$; more formally, we interpret $p$ in $\mathcal{S}(r_S)$ in such a way that for all $i_1, \ldots, i_{|p|}$ in the domain of $\mathcal{S}(r_S)$ the tuple $(i_1, \ldots, i_h)$ belongs to the interpretation of $p$ if and only if $\mathcal{T}(r_T) \models p(f^{-1}_{r_T, r_S}(i_1), \ldots, f^{-1}_{r_T, r_S}(i_h))$ holds.

We may then continue in this way: once we have assigned a $\tau$-interpretation $\mathcal{S}'(s)$ for $s \in S_h$ (with $h < n$) extending the $\tau \setminus \{p\}$-interpretation $\mathcal{S}(s)$, we proceed by assigning a $\tau$-interpretation $\mathcal{S}'(s')$ to all sons $s'$ of $s$: if $(t', s')$ is the unique pair in $B_{n-(h+1)}$ with second component equal to $s'$, we use the isomorphism $f_{s'} := f_{t',s'}$ to define, for all $i_1, \ldots, i_{|p|}$ in the domain of $\mathcal{S}(s')$,

$$\mathcal{S}'(s') \models p(i_1, \ldots, i_h) \Leftrightarrow \mathcal{T}(t') \models p(f_{s'}^{-1}(i_1), \ldots, f_{s'}^{-1}(i_h)).$$

Notice that the interpretations of $p$ are consistent with the tree structure imposed by $(S, \mathcal{S})$, that is: if $u, v$ are two nodes in $S$ whose distance from $r_S$ is at most $n$ and $v$ is a son of $u$, then the $\tau$-interpretations $\mathcal{S}'(u), \mathcal{S}'(v)$ agree on $\mathcal{S}(u, v)$; this is true because the inverse $f_u^{-1}, f_v^{-1}$ of the isomorphisms $f_u, f_v$ used to define the $p$ interpretation on $\mathcal{S}'(u), \mathcal{S}'(v)$, respectively, agree on $\mathcal{S}(u, v)$.

In order to complete the definition of the $\Sigma_{\tau',\tau}$ tree $(S, \mathcal{S}')$, we have to define $\mathcal{S}'(s)$ for all nodes $s$ belonging to level $S_h$ with $h > n$. For such an $s$ we consider the unique node $s_0 \in S_n$ which is the ancestor of $s$ and we assign to $\mathcal{S}(s)$ an interpretation of $p$ in such a way that the resulting $\tau$-structure $\mathcal{S}'(s)$ agree with $\mathcal{S}'(s_0)$ on $\mathcal{S}(s_0, s')$.

$\square$

## 5. Expressing bounded bisimulations

In this section we prove that for all $n$ the set of all $(\tau', \tau)$-structures which are $n$-bisimilar to a given structure is definable by a sentence of guarded first order logic.

In the proof of the next lemma we shall use the following notations, regarding a variable $x$ and tuples $\mathbf{a}, \mathbf{a}'$ of elements in a structure.

- $|\mathbf{a}|$ is the length of the tuple $\mathbf{a}$ and $\mathbf{x_a} := x_1, \ldots, x_{|a|}$;
- if $r$ is a relational symbol of arity $h$ then $|r| := (1, \ldots, h)$;
- if $i_1, \ldots, i_n$ is a tuple of indices, then $\mathbf{x}_{i_1,\ldots,i_n} := x_{i_1}, \ldots, x_{i_n}$;
- $ix(\mathbf{a}', \mathbf{a})$ is the tuple of indices from $\mathbf{a}'$, listed in increasing order, corresponding to elements in the tuple $\mathbf{a}'$ which belong to the tuple $\mathbf{a}$;
- $\mathbf{x}_{\mathbf{a}',\mathbf{a}} := \mathbf{x}_{ix(\mathbf{a}',\mathbf{a})}$;

- if $f$ is a function between tuples of indices with domain $\mathbf{i} = (i_1, \ldots, i_h)$ then $\mathbf{x}_f$ stands for $x_{f(i_1)}, \ldots, x_{f(i_h)}$;

- $f_{\mathbf{a}',\mathbf{a}}$ is the function with domain $ix(\mathbf{a}', \mathbf{a}) = (i_1, \ldots, i_n)$ and codomain $ix(\mathbf{a}, \mathbf{a}')$ defined by

$$f_{\mathbf{a}',\mathbf{a}}(i) = min\{j \in ix(\mathbf{a}, \mathbf{a}') : a'_i = a_j\};$$

- $ix(\mathbf{a}' \backslash \mathbf{a})$ is the tuple of indices, listed in increasing order, corresponding to elements in the tuple $\mathbf{a}'$ which do not belong to the tuple $\mathbf{a}$;

- $\mathbf{x}_{\mathbf{a}' \backslash \mathbf{a}} := \mathbf{x}_{ix(\mathbf{a}' \backslash \mathbf{a})}$;

- if $\mathbf{x} = x_1, \ldots, x_n$, $\mathbf{y} = y_1, \ldots, y_n$ are tuples of variables of the same length and $\phi$ is a formula, then $\phi[\mathbf{y} \leftarrow \mathbf{x}]$ denotes the formula obtained form $\phi$ by substituting $y_i$ for $x_i$.

Similarly, we have $\mathbf{y}_\mathbf{a} := y_1, \ldots, y_{|a|}$, and so forth. Let us give an example: consider $\mathbf{a} = u, v, w, t, u$, $\mathbf{a}' = b, w, w, u$; then

$$ix(\mathbf{a}', \mathbf{a}) = (2, 3, 4);\ ix(\mathbf{a}, \mathbf{a}') = (1, 3, 5);\ \mathbf{x}_{\mathbf{a}',\mathbf{a}} = x_2, x_3, x_4;$$

$$f_{\mathbf{a}',\mathbf{a}}(2) = f_{\mathbf{a}',\mathbf{a}}(3) = 3;\ f_{\mathbf{a}',\mathbf{a}}(4) = 1;\ \mathbf{y}_{f_{\mathbf{a}',\mathbf{a}}} = y_3, y_3, y_1.$$

Moreover, if $\alpha = r(\mathbf{x}_{\mathbf{a}'}) = r(x_1, x_2, x_3, x_4)$, then

$$\alpha[\mathbf{y}_{f_{\mathbf{a}',\mathbf{a}}} \leftarrow \mathbf{x}_{\mathbf{a}',\mathbf{a}}] = r(x_1, y_3, y_3, y_1).$$

**Lemma 5.1.** *For any $(\tau', \tau)$-structure $(\mathfrak{A}, \mathbf{a})$ and natural number $n$ there exists a $GF(\tau', \tau)$ formula $(\tau', \tau)^n(\mathfrak{A}, \mathbf{a})$ with $free((\tau', \tau)^n(\mathfrak{A}, \mathbf{a})) = \mathbf{x}_\mathbf{a}$ and quantification rank equal to $n$ such that for all $(\tau', \tau)$- structures $(\mathfrak{B}, \mathbf{b})$ it holds*

$$(\mathfrak{B}, \mathbf{b}) \models (\tau', \tau)^n(\mathfrak{A}, \mathbf{a}) \quad \Leftrightarrow \quad (\mathfrak{A}, \mathbf{a}) \sim^n (\mathfrak{B}, \mathbf{b}).$$

*Proof.* The formula $(\tau', \tau)^n(\mathfrak{A}, \mathbf{a})$ is defined by induction on $n$ as follows. For $n = 0$, $(\tau', \tau)^0(\mathfrak{A}, \mathbf{a})$ is

$$\bigwedge\{\alpha(\mathbf{x}_\mathbf{a}) : \alpha \text{ is atomic or negated atomic in } \tau \text{ and } (\mathfrak{A}, \mathbf{a}) \models \alpha(\mathbf{x}_\mathbf{a})\}.$$

In the induction step, we define $(\tau', \tau)^{n+1}(\mathfrak{A}, \mathbf{a})$ to be the conjunction of the formula $(\tau', \tau)^0(\mathfrak{A}, \mathbf{a})$ with two formulas. The first one accounts for the "forth" part of a bounded guarded bisimulation:

$$\bigwedge_{r \in \tau', \mathbf{a}' \in Guard_r(\mathfrak{A})} \exists \mathbf{y}_{\mathbf{a}' \backslash \mathbf{a}} \left[ (r(\mathbf{x}_{\mathbf{a}'}) \wedge (\tau', \tau)^n(\mathfrak{A}, \mathbf{a}'))[\mathbf{y}_{\mathbf{a}' \backslash \mathbf{a}} \leftarrow \mathbf{x}_{\mathbf{a}' \backslash \mathbf{a}}; \mathbf{x}_{f(\mathbf{a}',\mathbf{a})} \leftarrow \mathbf{x}_{\mathbf{a}',\mathbf{a}}] \right].$$

Notice the role of variable substitutions in the formula: if $\mathbf{a}'$ is guarded by $r$ in $\mathfrak{A}$, then we need to assert the existence of an $r$-guarded tuple $\mathbf{b}'$ in $\mathfrak{B}$ satisfying the formula $(\tau', \tau)^n(\mathfrak{A}, \mathbf{a}')$. However, the elements in $\mathbf{b}'$ corresponding to elements in $\mathbf{a}'$ which are also elements in $\mathbf{a}$ shouldn't be new. This is the reason for which we write $\exists \mathbf{y}_{\mathbf{a}'\setminus\mathbf{a}}$ in the formula, and we substitute $\mathbf{y}_{\mathbf{a}'\setminus\mathbf{a}}$ for $\mathbf{x}_{\mathbf{a}'\setminus\mathbf{a}}$ and $\mathbf{x}_{f(\mathbf{a}',\mathbf{a})}$ (the already known elements of $\mathbf{b}'$ ) for $\mathbf{x}_{\mathbf{a}',\mathbf{a}}$.

Similarly, the second formula in the conjunction accounts for the "back" part of a bounded guarded bisimulation:

$$\bigwedge_{\substack{r\in\tau' \\ \mathbf{i}\subseteq|r|, \\ f:\mathbf{i}\to\{1,\dots,|\mathbf{a}|\}}} \forall \mathbf{y}_{|r|\setminus\mathbf{i}} \left[ (r(\mathbf{x}_{|r|}) \to \bigvee_{\substack{\mathbf{a}'\in Guard_r(\mathfrak{A}) \\ f_{\mathbf{a}',\mathbf{a}}=f}} (\tau', \tau)^n(\mathfrak{A}, \mathbf{a}'))[\mathbf{y}_{|r|\setminus\mathbf{i}} \leftarrow \mathbf{x}_{|r|\setminus\mathbf{i}}; \mathbf{x}_f \leftarrow \mathbf{x}_\mathbf{i}] \right]$$

$\square$

The formula $(\tau', \tau)^n(\mathfrak{A}, \mathbf{a})$ is called the $n$-bisimulation type of the $(\tau', \tau)$-structure $(\mathfrak{A}, \mathbf{a})$.

## 6. Bisimulation Quantifiers for the Guarded Fragment

The main ingredient of a proof of uniform interpolation for $GF$ is the elimination of the so called "bisimulation quantifiers", which are non standard second-order quantifiers asserting the existence of subsets not necessarily in the domain of the model, but possibly in a bisimilar copy of it. This approach is inspired by [10] and [6], and has been successfully used to prove the uniform interpolation property for various extensions of modal logic, such as the $\mu$-calculus (see [2]).

**Definition 6.1.** *Given a formula $\phi \in Guard(\tau', \tau)$ and $p \in \tau \setminus \tau'$, we extend the syntax of guarded formulas with a new quantifier $\widetilde{\exists} p\ \phi$; the new formula is viewed as a formula in the $(\tau', \tau \setminus \{p\})$ language and it is evaluated over a $(\tau', \tau \setminus \{p\})$-structure $(\mathfrak{A}, \mathbf{a})$ as follows:*

$$(\mathfrak{A}, \mathbf{a}) \models \widetilde{\exists} p\ \phi \Leftrightarrow \textit{ there exists a } (\tau', \tau) \textit{ structure } (\mathfrak{B}, \mathbf{b}) \textit{ with}$$

$$(\mathfrak{B}, \mathbf{b}) \models \phi \textit{ and } (\mathfrak{A}, \mathbf{a}) \sim^{(\tau', \tau\setminus\{p\})} (\mathfrak{B}, \mathbf{b}).$$

Our goal is to prove $\widetilde{\exists} p$ elimination in $GF(\tau', \tau)$, for all $p \in \tau \setminus \tau'$. We start by eliminating $\widetilde{\exists} p$ in front of type formulas.

**Lemma 6.2.** *For any $(\tau', \tau)$-structure $(\mathfrak{A}, \mathbf{a})$ we have*

$$\widetilde{\exists} p[(\tau', \tau)^n(\mathfrak{A}, \mathbf{a})] \equiv (\tau', \tau \setminus \{p\})^n(\mathfrak{A}, \mathbf{a})$$

*Proof.* This is actually the content of Lemma 4.1, because using this Lemma we can prove that the semantics of $(\tau', \tau \setminus \{p\})^n(\mathfrak{A}, \mathbf{a})$ coincides with the one for $\widetilde{\exists} p[(\tau', \tau)^n(\mathfrak{A}, \mathbf{a})]$. For any $(\tau', \tau \setminus \{p\})$-structure $(\mathfrak{B}, \mathbf{b})$, suppose $(\mathfrak{B}, \mathbf{b}) \models (\tau', \tau \setminus \{p\})^n(\mathfrak{A}, \mathbf{a})$; then $(\mathfrak{A}, \mathbf{a}) \sim_n^{(\tau', \tau \setminus \{p\})} (\mathfrak{B}, \mathbf{b})$ and by Lemma 4.1 there exists a structure $(\mathfrak{C}, \mathbf{c})$ with

$$(\mathfrak{C}, \mathbf{c}) \sim_n^{(\tau', \tau)} (\mathfrak{A}, \mathbf{a}), \qquad (\mathfrak{C}, \mathbf{c}) \sim^{(\tau', \tau \setminus \{p\})} (\mathfrak{B}, \mathbf{b}).$$

We obtain $(\mathfrak{C}, \mathbf{c}) \models (\tau', \tau)^n(\mathfrak{A}, \mathbf{a})$, and from $(\mathfrak{C}, \mathbf{c}) \sim^{(\tau', \tau \setminus \{p\})} (\mathfrak{B}, \mathbf{b})$ it follows $(\mathfrak{B}, \mathbf{b}) \models \widetilde{\exists} p(\tau', \tau)^n(\mathfrak{A}, \mathbf{a})$.

Viceversa, if $(\mathfrak{B}, \mathbf{b}) \models \widetilde{\exists} p(\tau', \tau)^n(\mathfrak{A}, \mathbf{a})$ then there exists a structure $(\mathfrak{C}, \mathbf{c})$ with

$$(\mathfrak{C}, \mathbf{c}) \sim^{(\tau', \tau \setminus \{p\})} (\mathfrak{B}, \mathbf{b}) \text{ and } (\mathfrak{C}, \mathbf{c}) \models (\tau', \tau)^n(\mathfrak{A}, \mathbf{a}).$$

Then, since $(\tau', \tau)^n(\mathfrak{A}, \mathbf{a})$ implies $(\tau', \tau \setminus \{p\})^n(\mathfrak{A}, \mathbf{a})$ and this last formula is invariant under $(\tau', \tau \setminus \{p\})$-bisimulation, we obtain

$$(\mathfrak{B}, \mathbf{b}) \models (\tau', \tau \setminus \{p\})^n(\mathfrak{A}, \mathbf{a}).$$

□

**Theorem 6.3.** *For any finite relational language $\tau$, $\phi \in Guard(\tau', \tau)$ and $p \in \tau \setminus \tau'$ there exists a formula $\psi \in Guard(\tau', \tau \setminus \{p\})$ which is equivalent to $\widetilde{\exists} p\, \phi$; in other words, for all $(\tau', \tau \setminus \{p\})$-structures $(\mathfrak{A}, \mathbf{a})$ it holds:*

$$(\mathfrak{A}, \mathbf{a}) \models \psi \Leftrightarrow \text{ there exists a } (\tau', \tau)\text{-structure } (\mathfrak{B}, \mathbf{b}) \text{ with}$$

$$(\mathfrak{B}, \mathbf{b}) \models \phi \text{ and } (\mathfrak{A}, \mathbf{a}) \sim^{(\tau', \tau \setminus \{p\})} (\mathfrak{B}, \mathbf{b})$$

*Proof.* As it is easily seen via semantics, the formula $\widetilde{\exists} p(\phi \vee \psi)$ is equivalent to $\widetilde{\exists} p\phi \vee \widetilde{\exists} p\psi$. Moreover, since a formula of quantification rank equal to $n$ is equivalent to a disjunction of $n$-types, it is enough to prove the theorem for $n$-types, which is what we did in Lemma 6.2. □

## 7. Modal Uniform Interpolation for the Guarded Fragment

We have now almost all ingredients to perform a classical proof of uniform interpolation via bisimulation quantifiers. The last ingredient is "amalgamation", which has been already proved in [1].

**Proposition 7.1. (Amalgamation [1])**
*Let $\tau, \sigma$ be two finite vocabularies both containing $\tau'$. Let $\mathfrak{A}, \mathfrak{B}$ be a $(\tau', \tau)$ and a $(\tau', \sigma)$ structure, respectively. If*

$$(\mathfrak{A}, \mathbf{a}) \sim^{(\tau', \tau \cap \sigma)} (\mathfrak{B}, \mathbf{b})$$

*there exists a $(\tau', \tau \cup \sigma)$ structure $(\mathfrak{C}, \mathbf{c})$ with*

$$(\mathfrak{A}, \mathbf{a}) \sim^{(\tau', \tau)} (\mathfrak{C}, \mathbf{c}),$$
$$(\mathfrak{B}, \mathbf{b}) \sim^{(\tau', \sigma)} (\mathfrak{C}, \mathbf{c}).$$

**Theorem. (Uniform Modal Interpolation for $GF(\tau', \tau)$)**
*Let $\tau', \tau, \sigma$ be finite vocabularies with $\tau' \subseteq \sigma \subseteq \tau$. For any formula $\phi \in Guard(\tau', \tau)$ there exists a formula $\theta \in Guard(\tau', \sigma)$ such that*

1. $\models \phi \to \theta$;
2. *if $\psi \in Guard(\tau', \nu)$ for a vocabulary $\nu$ such that $\tau' \subseteq \nu$, $\tau \cap \nu \subseteq \sigma$, and $\models \phi \to \psi$, then $\models \theta \to \psi$.*

*Proof.* Let $\tau \setminus \sigma = \{p_1, \ldots, p_n\}$. By hypothesis, $p_i \notin \tau'$, for all $i$. By Theorem 6.3 we know that there exists a formula $\theta \in Guard(\tau', \sigma)$ which is equivalent to the formula $\widetilde{\exists} p_1 \ldots \widetilde{\exists} p_n \phi$. Consider a $(\tau', \tau)$-structure $(\mathfrak{A}, \mathbf{a})$ with $(\mathfrak{A}, \mathbf{a}) \models \phi$. Since $(\mathfrak{A}, \mathbf{a}) \sim^{(\tau', \tau \setminus \{p_n\})} (\mathfrak{A}, \mathbf{a})$ holds, we have $(\mathfrak{A}, \mathbf{a}) \models \widetilde{\exists} p_n \phi$. Proceeding in this way for all bisimulation quantifiers $\widetilde{\exists} p_i$, we obtain $(\mathfrak{A}, \mathbf{a}) \models \widetilde{\exists} p_1 \ldots \widetilde{\exists} p_n \phi$. This proves 1.

In order to prove 2., suppose we have $\psi \in Guard(\tau', \nu)$ with $\models \phi \to \psi$ and $\tau \cap \nu \subseteq \sigma$, and consider a $(\tau', \sigma \cup \nu)$-structure $(\mathfrak{A}, \mathbf{a})$ with $(\mathfrak{A}, \mathbf{a}) \models \theta$.

Then by definition of $\widetilde{\exists}$ there exists a $(\tau', \tau)$-structure $(\mathfrak{B}, \mathbf{b})$ with

$$(\mathfrak{B}, \mathbf{b}) \models \phi \text{ and } (\mathfrak{A}, \mathbf{a}) \sim^{(\tau', \sigma)} (\mathfrak{B}, \mathbf{b}).$$

Since $\tau \cap \nu \subseteq \sigma$, by Proposition 7.1, we obtain a $(\tau', \tau \cup \nu)$-structure $(\mathfrak{C}, \mathbf{c})$ with

$$(\mathfrak{A}, \mathbf{a}) \sim^{(\tau', \nu)} (\mathfrak{C}, \mathbf{c}),$$
$$(\mathfrak{B}, \mathbf{b}) \sim^{(\tau', \tau)} (\mathfrak{C}, \mathbf{c}).$$

Following the second bisimulation we obtain $(\mathfrak{C}, \mathbf{c}) \models \phi$; since $\psi$ is implied by $\phi$ we have $(\mathfrak{C}, \mathbf{c}) \models \psi$ and following the first bisimulation we obtain $(\mathfrak{A}, \mathbf{a}) \models \psi$. $\square$

## 8. Conclusions and further work

In this paper we proved a form of Uniform Interpolation for the Guarded Fragment, which we called *Modal* because of the attention that must be paid to the difference between *modalities* and *propositions*. The proof of this result relies heavily on bounded bisimulations, and in a future research we will explore the possibility of a similar proof for the Guarded Negation fragment of first order logic, where full Craig interpolation holds (see [9]). We will also consider the fixed point extension of the Guarded Fragment, and try to prove Modal (Uniform) Interpolation for this logic, although in this case we will not be able to use bounded bisimulation, and a proof using automata seems more appropriate.

## References


[1] E. Hoogland, M. Marx. Interpolation and Definability in Guarded Fragments. *Studia Logica* 70(3): 373–409, 2002.

[2] G. D'Agostino and M. Hollenberg. Logical questions concerning the $\mu$-calculus. *Journal of Symbolic Logic*, 65:310–332, 2000.

[3] G. D'Agostino and G. Lenzi. On Modal $\mu$-Calculus with Explicit Interpolants. *Journal of Applied Logic*, 4(3): 256-278, 2006.

[4] E. Grädel. On the restraining power of guards. *Journal of Symbolic Logic*, 64:1719-1742, 1999.

[5] E. Grädel. Decision Procedures for Guarded Logics. *Automated Deduction CADE-16*, Lecture Notes in Computer Science Volume 1632, 1999, pp 31-51.

[6] A. Pitts. On an interpretation of second-order quantification in first-order intuitionistic propositional logic. *Journal of Symbolic Logic*, 57:33–52, 1992.

[7] V. Bárány, B. ten Cate, and L. Segoufin. Guarded Negation. ICALP (2) 2011: 356-367.

[8] B. ten Cate. Guarded Negation. Stanford Logic Seminar 10/02/2012; http://www-logic.stanford.edu/seminar/1213/tenCate_oct2012.pdf.

[9] V, Bárány, M. Benedikt, and B. ten Cate, Rewriting Guarded Negation Queries, MFCS 2013: 98-110 http://dx.doi.org/10.1007/978-3-642-40313-2_11.

[10] A. Visser. Bisimulations, model descriptions and propositional quantifiers. Logic Group Preprint Series 161, Dept. of Philosophy, Utrecht University, 1996.